The Value of Money: Theory and Practice

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The problem of "normalizing" the value of economic data expressed in money units, in order to take into account the changing value of money, is an old one. It is surprising that a "good" solution is still to be found, although a number of "accepted" ones are used today in a variety of contexts. We propose to investigate here an approach that would appear to show a number of desirable characteristics. We will introduce the theory and later apply it to a specific interesting case.

The "value of money" is the *relative purchasing power* of the unit of money, i.e. the ability of the specified unit of money to purchase goods and services at a given time t_i *relative* to its ability to do so at some other time t_j. If there is a single commodity, quite clearly the value of money at t_i relative to its value at time t_j is determined straightforwardly by the ratio of the prices at the two different times. However, as soon as we have to deal with more than one commodity, the problem arises of how to "combine" the various prices into an overall measure. This implies obviously that the "value of money" can only be stated *relative to a specific set of commodities*. There has been considerable theoretical work on the issue, but the practical implementation of many ideas appears to have been lagging. W. E. Diewert and A. O. Nakamura (1993) provide a comprehensive review of the issue and an extensive bibliography.

I. Overview of the Current Situation

We will not even remotely attempt to review the current state of the theory and practice, but we will focus on a few specific issues. In all that follows we assume that for any given time interval t_i there is an *average price*, p_{im} for each commodity, *m,* and that the price is *predetermined*. In other words, we will not investigate in any way the factors that may affect the price of any commodity. We will only be concerned with the effect that the varying prices have on the *purchasing power* of the *money unit* at any given time. We assume that all transactions for each commodity during a given time interval occur at the average price for that interval. We will denote by q_{im} the quantity of each commodity purchased at time t_i.
A methodology that has received considerable attention from both the theoretical and practical point of view is the one based on *Fisher "ideal" price indexes*. Diewert (1987)

provides a complete review from both the historical and theoretical points of view. The Fisher price index for time t_i relative to time t_j, $F(i,j)$ is defined as

(1) $$F(i,j)=\sqrt{\left(L(i,j)\times P(i,j)\right)}$$

where

(2) $$L(i,j)=\frac{\sum_m p_{im}q_{jm}}{\sum_m p_{jm}q_{jm}}$$

is the *Laspeyres* price index and

(3) $$P(i,j)=\frac{\sum_m p_{im}q_{im}}{\sum_m p_{jm}q_{im}}$$

is the *Paasche* price index. There has been considerable attention paid in the literature to the fact that the Fisher price index satisfies a number of a priori "tests" that have been put forth by a number of people, supposedly characterizing what a "good" index should do. Unfortunately, the Fisher price index fails to satisfy a number of other tests, both theoretical and pragmatical. In general the Fisher index does *not* satisfy the relation

(4) $$F(i,j)=F(i,k)F(k,j)$$

for arbitrary *i*, *j* and *k*. In other words it does not have the *transitivity property* (that in some economics literature has been sometimes referred to as the "circularity property"). This could be "fixed" by selecting a specific time interval, t_0 as the *reference time interval* and define the *relative* Fisher price index, $F_r(i,j)$ as

(5) $$F_r(i,j)=F(i,0)/F(j,0)\,.$$

However, because of lack of the transitivity property, the results would actually depend on what time interval would be selected as reference, making the whole process essentially meaningless. In addition, the Fisher price index has a conceptual difficulty in handling "new" commodities or "obsolete" commodities. If a commodity appears in the set of commodities for some time interval, t_i, but not for another time interval, t_j, it is not possible to compute the value of the Fisher index for those times since there is no price associated with the commodity at the time it is absent. The Fisher index is also inadequate in the way it handles the issue of *commodity substitution* associated with the drastic change in the relative prices of some commodities. An example will clarify the situation. Let there be only two commodities and let the prices and quantities for two time intervals t_1 and t_2 be

(6) $$p_{11}=1; \quad p_{12}=1; \quad q_{11}=1; \quad q_{12}=1;$$

(7) $$p_{21}=1; \quad p_{22}=10000; \quad q_{21}=2; \quad q_{22}=0;$$

The corresponding value of the Fisher price index is

(8) $$F(2,1)=\sqrt{\frac{10001\times 2}{2\times 2}}\cong 70.7$$

indicating that the "average price level" for t_2 relative to t_1 is seventy times higher, a result that does not appear acceptable (although what the "right" result "ought" to be is by no means obvious). The problems of the "missing commodity" and of drastic changes in prices are obviously more likely to occur the further the time intervals are separated in time, therefore making the choice of a fixed reference time interval less palatable in the context of a long time series.

In order to deal with this problem the notion of the Fisher *chain* price indexes has been introduced. There are two assumptions on which the idea is based, namely:

1. whenever a "new" commodity appears, at its first appearance it normally has a low quantity value associated with it and therefore it represents a small portion of the overall total purchasing value of all commodities (a symmetric argument exists for an "obsolete" commodity);
2. price changes occur gradually, so that the difference in relative commodity prices is never very large for time intervals closely spaced in real time.

We will refer to the Fisher price index defined in (1) as the Fisher standard price index. In the context of a series of time intervals, t_i, with i=1,2,….,N, we define the Fisher chain price index,
relative to a pre-designated time interval t_j, $F_c(i,j)$, recursively as follows:

(9) $$F_c(j,j)=1$$

(10) $$F_c(j+k,j)=F(j+k,j+k-1)F_c(j+k-1,j)$$

(11) $$F_c(j-k,j)=F(j-k,j-k+1)F_c(j-k+1,j)$$

for $k\geq 1$.This implies that the Fisher standard price index needs only to be evaluated for consecutive time periods. If there is a "missing" commodity, we can deal with the problem by limiting the evaluation of the base index only to those commodities that are in common. In view of assumption 1, this will produce a negligible distortion. The Fisher chain index satisfies the transitivity property, i.e.

(12) $$F_c(i,k)=F_c(i,j)F_c\left(j,k\right)$$

How good is the new procedure at dealing with rapidly changing prices? Consider again the previous numerical example and assume that there are three intervening time periods, with the overall picture being as follows:

(13) $$p_{11}=1; \quad p_{12}=1; \quad q_{11}=1; \quad q_{12}=1;$$

(14) $$p_{21}=1; \quad p_{22}=10; \quad q_{21}=1.5; \quad q_{22}=0.05;$$

(15) $$p_{31}=1; \quad p_{32}=100; \quad q_{31}=1.8; \quad q_{32}=0.002;$$

(16) $$p_{41}=1; \quad p_{42}=1000; \quad q_{41}=1.9; \quad q_{42}=0.0001;$$

(17) $$p_{51}=1; \quad p_{52}=10000; \quad q_{51}=2; \quad q_{52}=0;$$

We have

(18) $$F(2,1)\cong 2.66; \quad F(3,2)\cong 1.89; \quad F(4,3)\cong 1.41; \quad F(5,4)\cong 1.20;$$

(19) $$F_c(5,1)\cong 8.51; \qquad F(5,1)\cong 70.7$$

showing that the new procedure certainly helps in reducing the value of the price index for the extreme time points to a more reasonable value. However, the change appears to be one of degree, since the resulting value of $F_c(5,1)$ still appears to be "abnormally high".

There is, however, a major conceptual problem with the Fisher chain price indexes. Assume that at times t_i and t_j the prices for all the commodities are identical. If t_i and t_j are consecutive, (i.e. $i = j + 1$), we have that

(20) $$F_c(i,j)=1$$

as it should be. However, if there is exactly one intermediate point, (i.e. $i = j + 2$), equation (20) will *not* hold in general *unless* also all of the quantities are the same for the two time intervals. If there are two or more intermediate points, (i.e. $i = j+k; \ k \geq 3$), then equation (20) will *not* hold in general even if the quantities are also the same. For example, let's assume that in the previous example the values for both prices and quantities at time t_5 are the same as for time t_1 . The value of $F_c(5,1)$ will be 0.31, rather than 1!

More generally, $F_c(i,j)$ for $i > j+1$, will depend on the value of any of the intermediate time intervals between t_i and t_j. This means, among other things, that if we keep t_i and t_j fixed and change the number of time intervals between them for which we accumulate data (e.g. we change from yearly to quarterly data), the value of $F_c(i,j)$ will in general change. Such anomalies would appear to be unacceptable.

We have discussed at some length the Fisher price indexes for one primary reason. The Bureau of Economic Analysis (BEA) of the US Department of Commerce has the responsibility to evaluate the "real" value of the GDP and other related quantities on behalf of the US Government. It has chosen to use the Fisher chain price index methodology as its primary methodology (Landefeld (1997)), although in certain contexts it also occasionally refers to the Fisher standard price indexes, relative to selected years, for short time spans around the reference year. We believe that there are better choices and the major intent of this paper is to demonstrate one such alternative.

II. The Basic Framework

The basic idea of the approach that we will discuss was put forth in 1924 by A. A. Konüs (1924). Although the methodology proposed by Konüs has been extensively discussed in the literature, it appears not to have been applied in practice, for reasons that we may be able to clarify later on.

We will consider the problem from the point of view of a *single individual purchaser* of goods and services. We assume that there are M commodities, with prices p_{im} at time t_i and for commodity *m*. We will denote in general a vector of prices by the symbol $\mathbf{p}$ and the value of the vector at time t_i by $\mathbf{p_i}$. We assume that at each time t_i the individual has a non-zero *budget*, b_i which the individual *will* spend on the purchase of a selection of quantities of the different available commodities. We will denote by $\mathbf{q}$ a general vector of commodity quantities. Let q_{im} denote the quantity of commodity *m* purchased at time t_i and let $\mathbf{q_i}$ denote the vector of quantities at time t_i. Obviously we must have

(21) $$b_i = \sum_m p_{im} q_{im} > 0$$

with the constraints

(22) $$q_{im} \geq 0 \qquad \text{for all } i \text{ and for all } m$$

which, in view of (21), implies that at least one of the q_{im} is strictly greater than zero. More succinctly,

(23) $$b_i = \mathbf{p_i} \bullet \mathbf{q_i}$$

(24) $$\mathbf{q_i} > \mathbf{0}$$

where we have used the notation "$\mathbf{x} \bullet \mathbf{y}$" to indicate the scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$ and the notation "$\mathbf{x} > \mathbf{0}$" to indicate that each component of the vector $\mathbf{x}$ is non-negative *and* that at least one of its component is strictly greater than zero.

We assume that there is a *utility function* $U(\mathbf{q})$ that determines for each individual the "level of satisfaction" achieved by the individual as a function of the quantity vector $\mathbf{q}$ purchased by the individual. We will assume for the time being that the function does not depend explicitly on time. We assume that the function $U(\mathbf{q})$ is a non-negative real valued function and that its domain is the set of all vectors $\mathbf{q}$ such that $\mathbf{q} \geq \mathbf{0}$. We will assume that the function $U(\mathbf{q})$ satisfies certain "regularity" properties, namely:

1. the function is *everywhere* continuous and differentiable with respect to every component of **q;**
2. for all values of $\mathbf{q}$ **:**

$$\frac{\partial U}{\partial \mathbf{q}} > \mathbf{0}$$

i.e. all first order derivatives are non-negative and at least one of them is strictly positive. These conditions maybe more stringent than strictly necessary, but this is not an issue, as it will become clearer in the following. The *utility* enjoyed by the purchaser at time t_i is given by

(25) $$u_i = U(\mathbf{q_i})$$

We assume that at any given time the purchaser *attempts* to acquire the various commodities in quantities such that the function u_i is maximized, subject to the conditions

(26) $$\mathbf{p_i} \bullet \mathbf{q_i} = b_i \qquad \mathbf{q_i} > \mathbf{0}$$

Define the *optimum utility function* $U^*(b, \mathbf{p} \,|\, U)$ as

(27) $$U^*(b, \mathbf{p} \,|\, U) = \max_{\mathbf{q}} \{U(\mathbf{q}) : \mathbf{p} \bullet \mathbf{q} = b, \mathbf{q} \geq 0\}$$

$U^*(b, \mathbf{p} \,|\, U)$ measures the *maximum* value of the utility function $U(\mathbf{q})$ that can be achieved by somebody with a budget *b*, given the price vector $\mathbf{p}$ and according to the utility function *U***.** Given the regularity conditions we have assumed, the function $U^*(b, \mathbf{p} \,|\, U)$ is strictly monotonically increasing with respect to *b*, for any $\mathbf{p}$. Note that for any positive α

(28) $$U^*(\alpha b, \alpha \mathbf{p} \,|\, U) = U^*(b, \mathbf{p} \,|\, U)$$

i.e. if both the price vector and the budget are multiplied by the same constant, the optimal value of the utility function remains invariant. Let's also define the *minimum budget function* $B^*(u, \mathbf{p} \,|\, U)$ as

(29) $$B^*(u,\mathbf{p}\,|\,U)=\min_{\mathbf{q}}\{\mathbf{p}\bullet\mathbf{q}:U(\mathbf{q})=u,\mathbf{q}\geq\mathbf{0}\}$$

In other words $B^*(u,\mathbf{p}\,|\,U)$ represents the *minimum* value of budget that is required, given the price vector $\mathbf{p}$, to purchase a quantity vector $\mathbf{q}$ that will achieve the value *u* for the utility function *U*. Given the regularity conditions that we have assumed, the function $B^*(u,\mathbf{p}\,|\,U)$ is strictly monotonically increasing with respect to *u*, for any $\mathbf{p}$. Note that for any positive α

(30) $$B^*(u,\alpha\mathbf{p}\,|\,U)=\alpha B^*(u,\mathbf{p}\,|\,U)$$

i.e. if the price vector is multiplied by a positive number, the budget necessary to achieve a specific value of the utility function is also multiplied by the same constant. It should be obvious that the functions $U^*(b,\mathbf{p}\,|\,U)$ and $B^*(u,\mathbf{p}\,|\,U)$ are the inverse of each other, i.e.:

(31) $$B^*(U^*(b,\mathbf{p}\,|\,U),\mathbf{p}\,|\,U)=b$$

(32) $$U^*(B^*(u,\mathbf{p},U),\mathbf{p}\,|\,U)=u$$

We can define the *price index function* $I(\mathbf{p_x},\mathbf{p_y},u\,|\,U)$ for the price vector $\mathbf{p_x}$ relative to the price vector $\mathbf{p_y}$, at the utility level *u*, as

(33) $$I(\mathbf{p_x},\mathbf{p_y},u\,|\,U)=\frac{B^*(u,\mathbf{p_x}\,|\,U)}{B^*(u,\mathbf{p_y}\,|\,U)}$$

The function $I(\mathbf{p_x},\mathbf{p_y},u\,|\,U)$ measures the ratio of the budgets required at the two specified price levels to achieve the same value *u* of the utility function, given an *optimal* selection of the corresponding quantity vectors. Note that for any $\mathbf{p_x}$, $\mathbf{p_y}$ and $\mathbf{p_z}$ we always have

(34) $$I(\mathbf{p_x},\mathbf{p_y},u\,|\,U)=I(\mathbf{p_x},\mathbf{p_z},u\,|\,U)I(\mathbf{p_z},\mathbf{p_y},u\,|\,U)$$

i.e. the *transitivity property* is always satisfied. Note also that

(35) $$I(\mathbf{p_x},\mathbf{p_y},u\,|\,U)=\frac{1}{I(\mathbf{p_y},\mathbf{p_x},u\,|\,U)}$$

showing that the defined function satisfies the *reciprocity property* at a *specific level of utility value u*. Finally we obviously have for all $\mathbf{p_x}$ and for all *u* that

(36) $$I(\mathbf{p_x},\mathbf{p_x},u)=1$$

implying that the value of the index for any two times at which the price vector is the same is always identically equal to 1, i.e. that the function $I(\mathbf{p_x},\mathbf{p_y},u\,|\,U)$ satisfies the *identity* property. Furthermore, the value of the index function does not depend in any way on the *times* at which the specified price vectors occur, but depends only on the value of those price vectors.

We can also define the *equivalent budget function* $B_e(\mathbf{p_x},\mathbf{p_y},b\,|\,U)$ as

(37) $$B_e(\mathbf{p_x},\mathbf{p_y},b\,|\,U) = B^*(U^*(b,\mathbf{p_y}\,|\,U),\mathbf{p_x}\,|\,U)$$

In other words, the function $B_e(\mathbf{p_x},\mathbf{p_y},b\,|\,U)$ measures the minimum budget that would be required, at the price vector $\mathbf{p_x}$, to achieve the same level of utility that could be *optimally* obtained with the budget *b* at the price vector $\mathbf{p_y}$, given the utility function *U***.** Notice that in general we have no reason to expect that the function $B_e(\mathbf{p_x},\mathbf{p_y},b\,|\,U)$ is linear with respect to *b*, for a given pair of price vectors, $\mathbf{p_x}$ and $\mathbf{p_y}$. In other words, in general it will be the case that

(38) $$B_e(\mathbf{p_x},\mathbf{p_y},\alpha b\,|\,U) \neq \alpha B_e(\mathbf{p_x},\mathbf{p_y},b\,|\,U)$$

We will define also the *equivalent relative budget function* $B_{er}(\mathbf{p_x},\mathbf{p_y},b\,|\,U)$ as

(39) $$B_{er}(\mathbf{p_x},\mathbf{p_y},b\,|\,U) = B_e(\mathbf{p_x},\mathbf{p_y},b\,|\,U)/b$$

Obviously we have

(40) $$B_{er}(\mathbf{p_x},\mathbf{p_y},b\,|\,U) = I(\mathbf{p_x},\mathbf{p_y},U^*(b,\mathbf{p_y})\,|\,U)$$

Consider the case of two time intervals t_i and t_j, with the corresponding values of the budgets and price vectors, b_i, $\mathbf{p_i}$ and b_j, $\mathbf{p_j}$. Let

(41) $$u_i = U^*(b_i,\mathbf{p_i}\,|\,U)$$

(42) $$u_j = U^*(b_j,\mathbf{p_j}\,|\,U)$$

(43) $$G_i = I(\mathbf{p_i},\mathbf{p_j},u_i\,|\,U)$$

(44) $$G_j = I(\mathbf{p_i},\mathbf{p_j},u_j\,|\,U)$$

In other words, G_i is the index evaluated at u_i, the (optimum) value of the utility at time t_i and G_j is the index evaluated at the (optimum) value of the utility function at time t_j. In general the two indexes will be different, unless it happens that $u_i = u_j$.

We should note here a clear analogy with the two components of the Fisher standard price Index, as discussed in Section I.. The Laspeyres index (that uses the quantities of the "base" time period) corresponds to the G_j index, while the Paasche index (that uses the quantities of the "target" time period) corresponds to the G_i. There is however a fundamental conceptual difference. In the case of the Fisher Index, the two components combine the *prices* at one time with the *quantities* at another. This has a fundamental weakness. If there is a drastic change in the price of a specific commodity (relative to the general change in price levels), the combination of the quantities purchased at one time at a relatively low price, with the high price at the other time, will produce an unreasonable effect, as already discussed in the first section. In the case of G_i and G_j, the two estimates for the relative value of money are based on attempting to achieve different values of the utility function. In each case, however, the prices at any one time are used to determine the *optimum choice of commodity quantities* for the different budget level. If there is a drastic increase in the relative price level of certain commodities, the optimization procedure will normally keep the quantity of those commodities at a lower level (possibly at zero) and will achieve the desired value of the utility function through a redistribution of the budget among the other commodities. In general there should be no significant distortion.

We could then define an *average* index $G(i, j)$ as the geometric average of the two indexes G_i and G_j i.e. as

(45) $$G(i, j) = \sqrt{G_i G_j}$$

that could be viewed as a "generalized" analog of the Fisher Index defined in equation (1).

We need here to open a parenthesis and discuss the traditional notion of "price index". The basic traditional *bilateral* problem has been to find a way of comparing the situation at two different times t_1 and t_2, under the assumption that at those times we have observed the price levels $\mathbf{p_1}$, $\mathbf{p}_2$ and the *actually purchased* quantity vectors $\mathbf{q_1}$, $\mathbf{q}_2$. It has been typically assumed that the "price index" for those two times "ought" to be a function not only of the price levels, i.e. of $\mathbf{p_1}$ and $\mathbf{p}_2$, but also of the *observed* quantity levels, $\mathbf{q_1}$ and $\mathbf{q}_2$. This is typically demonstrated by the definition of the Fisher standard price index, as given in equation (1). In the analysis we have pursued so far there has been no reference to the quantities actually purchased at any given time. However, we have used indirectly the notion of the quantities that *ought to have been purchased* in order to achieve the minimum budget at a given utility level or the maximum utility at a given budget level.

Traditionally, in most analyses based on an underlying utility function, it has been assumed that the quantities purchased at any given price level would be the ones that *actually achieved the maximum value* of the underlying utility function. We said earlier that we assume that the purchaser of goods and services *attempts* to maximize the utility function, not that it actually *succeeds*. What would stop our purchaser from achieving the desired optimization? There are at least two different perspectives.

The first is that in reality the purchaser does not operate in a market of unlimited and unconstrained resources. The desire of the purchaser to acquire a certain quantity of a certain commodity may be limited by the actual availability of that commodity. Also, it is possible that the acquisition of a certain amount of a given commodity may be linked to the purchase of some other commodity, therefore constraining the relative ratios in which certain commodities may be purchased. An even more important issue is that a purchaser does not typically observe *all* prices before making any single acquisition, a procedure that would be required in order to guarantee that the optimum quantity of each commodity is actually selected. Finally, the purchaser may make random mistakes.

But the more fundamental issue, from our perspective, is that the assumption of a purchaser behavior based on the maximization of a utility function is only a *convenient mathematical framework*. The purpose of the model is to *approximate* as closely as possible the actual observed behavior. No one really believes that a real life purchaser would go around measuring all available prices and then would sit down with a computer to determine what quantities should be bought. To require an *exact match* between the values of the *actually purchased* quantities and the values predicted by the model is unnecessary.

III. The Basic Model

All of the above is rather straightforward and was presented mainly to establish the conceptual and notational framework.. In order to transform the above set of definitions into a workable procedure we must choose a specific expression for the function $U(\mathbf{q})$. Whenever one has to select a specific mathematical model to represent a class of real world phenomena it is rather common to proceed in two phases. In the first phase we select a *class* of mathematical models that we believe to be appropriate for the specific phenomena. Such class is typically characterized by a vector of parameters. In the second phase we proceed to select a specific value of the parameter vector on the basis of observed data.

We assume that the function is $U(\mathbf{q})$ of the form

(46) $$U(\mathbf{q}) = \sum_m \frac{a_m c_m q_m}{c_m + q_m}$$

where the a_m and c_m are constant parameters associated with each commodity *m* satisfying the conditions

(47) $$a_m > 0 \quad \text{and} \quad c_m > 0 \qquad \text{for all } m$$

We will use the vector notation "**a**" and "**c**" to indicate the vectors whose components are the a_m and c_m parameters respectively. Note that the vectors **a**, **c**, **p** and **q** have all the same dimensionality, i.e. the number M of commodities. The function U is obviously completely characterized by the pair of vectors, **a** and **c**. Note that the function U is continuous, differentiable and that for all values of **q,**

(48) $$\frac{\partial U}{\partial q_m} = \frac{a_m c_m^2}{(c_m + q_m)^2} > 0$$

The function $U(\mathbf{q})$ achieves its finite maximum of

(49) $$\sum_m a_m c_m$$

when the quantities are infinite. Since the maximization of the function $U(\mathbf{q})$ is unaffected by multiplying it by a constant positive factor, we can assume, without any loss of generality, that the function U is *normalized* so that

(50) $$\sum_m a_m c_m = 1$$

or, more concisely,

(51) $$\mathbf{a} \bullet \mathbf{c} = 1$$

This implies that when the values of the quantities q_m are *infinite*, the function U achieves its maximum value of 1. What is the "meaning" of the constants a_m's and c_m's? The maximum contribution to the function $U(\mathbf{q})$ that any given commodity m may provide, when it is infinite, is obviously $a_m c_m$. If $q_m = c_m$, the contribution of that particular commodity to the overall value of the function will be $a_m c_m / 2$, i.e. exactly one half of its possible maximum. When the value of q_m is small compared with c_m, the contribution due to commodity m will be approximately $a_m q_m$, i.e. the parameter a_m represents the initial value of the derivative of $U(\mathbf{q})$ with respect to q_m, i.e. the initial rate of "desirability" for commodity m.

Before we continue we need to discuss the "reasonableness" of the assumptions we just made. The idea that the "consumer" behaves so as to maximize some kind of "utility function" is obviously an old one. Is our specific choice of function a reasonable one?

Strangely enough, it has been common in the economic literature on the subject of price indexes to talk about utility functions that are *linear* with respect to the vector $\mathbf{q}$, i.e. it has often been assumed that, for any positive constant α

$$U(\alpha\mathbf{q}) = \alpha U(\mathbf{q}) \tag{52}$$

This appears to us to be extremely strange. The most "natural" assumption about a "reasonable" utility function would appear to be that the incremental value of increasing the value of all quantities by a given amount should be a *decreasing function* of the value of the quantities. By analogy with a different class of economic analysis, it would appear reasonable to presume a "law of diminishing utility", conceptually similar to the well-known "law of diminishing returns". If the "utility" of owning a car is x, it is difficult to believe that the "utility" of owning 100 cars is $100x$. Our choice of functional class is such that the derivative with respect to q_m is monotonically decreasing with respect to q_m. This is not necessarily a strong requirement. It is conceivable that for some commodities the derivative with respect to q_m could be *increasing* for "small" values of q_m; however, it would appear that for "large" values of q_m, the derivative "ought" to be decreasing as q_m increases. Our choice of function allows for *unlimited substitutability*, i.e. any commodity can be substituted for another in order to achieve a given value of $U(\mathbf{q})$. It is quite clear that some level of substitutability is a necessary characteristic of any acceptable utility function. It may be argued that *too much substitutability* is not appropriate, since in practice some commodities can only be substituted by a restricted set of other commodities. However, we believe that there must be some compromise between simplicity and accuracy and that our choice is an adequate one for the purpose at hand.

Let

$$H(\mathbf{q},\mathbf{p},b) = U(\mathbf{q}) - \lambda^2\left(\sum_m p_m q_m - b\right) \tag{53}$$

The optimal choice for the q_m's are the values that satisfy the relations

$$\frac{\partial H(\mathbf{q},\mathbf{p},b)}{\partial q_m} = \frac{\partial U(\mathbf{q})}{\partial q_m} - \lambda^2 p_m = 0 \tag{54}$$

subject to the condition $\mathbf{q} > \mathbf{0}$. This implies that the optimum quantities q_m^* must satisfy the relations

$$\frac{a_m c_m^2}{(c_m + q_m^*)^2} = \lambda^2 p_m \tag{55}$$

from which we can derive the optimum values of the quantities to be

(56) $$q_m^*(b,\mathbf{p}\,|\,\mathbf{a},\mathbf{c}) = c_m\left[\frac{1}{\lambda(b)}\sqrt{\frac{a_m}{p_m}} - 1\right]$$

where

(57) $$\lambda(b) = \frac{\sum_m c_m\sqrt{a_m p_m}}{b+\sum_m c_m p_m}$$

in order to satisfy the condition

(58) $$\mathbf{p}\bullet\mathbf{q} = b$$

However, it may result that the quantities $q_m^*(b,\mathbf{p}\,|\,\mathbf{a},\mathbf{c})$ from equation (56) are negative. This obviously violates condition the condition $\mathbf{q} > \mathbf{0}$. The solution is that the quantities of such commodities must be set to zero and the corresponding index must be removed from the summation of equation (57). Note that, for any positive α we will have

(59) $$q_m^*(\alpha b,\alpha\mathbf{p}\,|\,\mathbf{a},\mathbf{c}) = q_m^*(b,\mathbf{p}\,|\,\mathbf{a},\mathbf{c})$$

In other words, if all prices changes at the same rate, the value of the quantities will remain the same *if* the budget is also increased by the same factor. By substituting from (56) and (57) into (46) we can derive the form of the function $U^*(b,\mathbf{p}\,|\,\mathbf{a},\mathbf{c})$, corresponding to definition (27), as

(60) $$U^*(b,\mathbf{p}\,|\,\mathbf{a},\mathbf{c}) = \sum_m a_m c_m - \frac{\left(\sum_m c_n\sqrt{a_m p_m}\right)^2}{b+\sum_m c_m p_m}$$

However, it should be remembered that all summations in equation (60) must be extended *only* to those commodities *m* for which equation (56) leads to non-negative values for the quantities $q_m^*(b,\mathbf{p}\,|\,\mathbf{a},\mathbf{c})$. We can also *formally* solve for the function $B^*(u,\mathbf{p}\,|\,\mathbf{a},\mathbf{c})$, corresponding to equation (29), as follows

(61) $$B^*(u,\mathbf{p}\,|\,\mathbf{a},\mathbf{c}) = \frac{\left(\sum_m c_m\sqrt{a_m p_m}\right)^2}{\sum_m a_m c_m - u} - \sum_m c_m p_m$$

We must remember again that all the summations must be extended *only* to those commodities for which equation (56) provides a non-negative result. Since this depends on the value of the budget, *b*, for which the quantities are estimated, the value of the function $B^*(u,\mathbf{p}\,|\,U)$ must be determined iteratively. In practice, the value given by equation (61), when the summations are extended to all commodities, provides a very good initial estimate. Typically, for large values of *b*, which correspond to large values of *u*, all commodities will have non-zero quantity values and equations (60) and (61) can be evaluated straightforwardly. However, at the other extreme, for very small values of *b*, which correspond to low values of *u*, the situation is quite different. For values of *b* near zero, only the commodity with the most favorable relation between initial "desirability" and "price" will have a non-zero quantity. This is determined by the factor

(62) $$\frac{a_m}{p_m}$$

having the highest value.

Assume that at some time t_i we know the values of b_i, of the prices vector $\mathbf{p_i}$ and of the quantity vector $\mathbf{q_i}$. Can we solve for $\mathbf{a}$ and $\mathbf{c}$? The answer is "yes", but not uniquely. The function $U(\mathbf{q})$ is characterized by 2M-1 parameters (since equation (50) removes one degree of freedom). However, at each time t_i we only have M-1 independent constraints, since condition (58) also removes one degree of freedom. If we know the values b_i, $\mathbf{p_i}$, $\mathbf{q_i}$ and b_j, $\mathbf{p_j}$, $\mathbf{q_j}$ for *two* time intervals t_i and t_j, we can then "almost" solve for the **a** and **c**, there being only one degree of freedom left. However, it is not true that we can find a solution for *any possible combination* of the above observed values. There are certain combinations that cannot be achieved for *any* selection of **a** and **c**. But finding a "solution" for **a** and **c** in such a way is not what we will try to do. As we mentioned earlier, we assume that the purchaser *attempts* to maximize the value of the utility function $U(\mathbf{q})$, but *not* that it actually *achieves* it. This means that we do not necessarily assume that the quantity vector $\mathbf{q_i}$ *actually acquired* by the purchaser at time t_i actually achieves the optimum value of the utility function, as determined by equation (60), for the appropriate value of the purchaser budget, b_i. What we *do* assume is that the purchaser selects the quantities of all commodities in such a way as to approximate as much as possible the optimal values appropriate to its budget and according to its utility function. In order to translate this assertion into an evaluation procedure for the parameter vectors, we can define an *error function* $S(\mathbf{a},\mathbf{c})$ that measures the "error" between the actual value of the quantity vector and its optimal value. There are a number of reasonable choices for such error function. We have chosen to use the one defined by equation (63), namely

(63) $$S(\mathbf{a},\mathbf{c}) = \sum_{i=1}^{N} \frac{1}{b_i^2} \left[\sum_{m=1}^{M} \left[p_{im} \left[q_{im} - q_m^*\left(b_i, \mathbf{p_i} \mid \mathbf{a},\mathbf{c}\right) \right] \right]^2 \right]$$

where the sum on *i* are extended over all times t_i for which data is available**.** The function $S(\mathbf{a},\mathbf{c})$ essentially measures the difference between the actual quantities purchased and the optimum quantities that would have been purchased, with the specified budget and the given price vector, if the given utility function would have been applied. The differences are weighed according to their percentage contribution to the total budget. Minimizing the function $S(\mathbf{a},\mathbf{c})$ is equivalent to attempt to find the pair of vectors $\mathbf{a}$ and $\mathbf{c}$ that will define a utility function that will determine a selection of quantities to be purchased that is as close as possible to those actually purchased by the consumer.

The problem of the actual minimization of the function $S(\mathbf{a},\mathbf{c})$ is a non-trivial one. The complexity of the function is such that there is no obvious way to solve the minimization problem algebraically. In any specific case, it is of course possible to solve the problem numerically, by a standard gradient procedure. There are, however, a number of practical problems. The complexity of the numerical computation to evaluate the value of the function $S(\mathbf{a},\mathbf{c})$ and of its gradient relative to $\mathbf{a}$ and $\mathbf{c}$ is approximately proportional to NM^2. In most interesting cases the value of M may be over 100. This will make the amount of computation non trivial. But this is not the most difficult issue. The major problem is that the minimization of a function of 2M-1 variables, with M in the hundreds, presents a number of pitfalls. A normal gradient procedure will lead to a *local minimum*, which is a function of the choice of the initial starting point. Any attempt to systematically search for a *global minimum* would entail a number of function evaluations of the order of $2^{(2M-1)}$. For M in the hundreds, this is impractical. The choice of "good" starting points and the decision to stop searching will have to be based on pragmatic considerations.

IV. The Choice of "Reference" Utility Values

Let's assume that we have chosen a pair of vectors, $\mathbf{a^o}$ and $\mathbf{c^o}$ which define our assumed utility function. We can now evaluate the price index function $I(\mathbf{p_x},\mathbf{p_y},u \mid U)$ for any pair of price vectors. However, we have to determine at what value *u* for the utility function we are going to make the evaluation, or, if necessary, how we should combine evaluations made at different utility levels. In Section II we introduced the two indexes, G_i and G_j that would appear to be good candidates if we had to compare only two price vectors (the *bilateral* problem). However, our approach must be able to be generalized to the case of an arbitrary long sequence of price vectors (the *multilateral* problem).

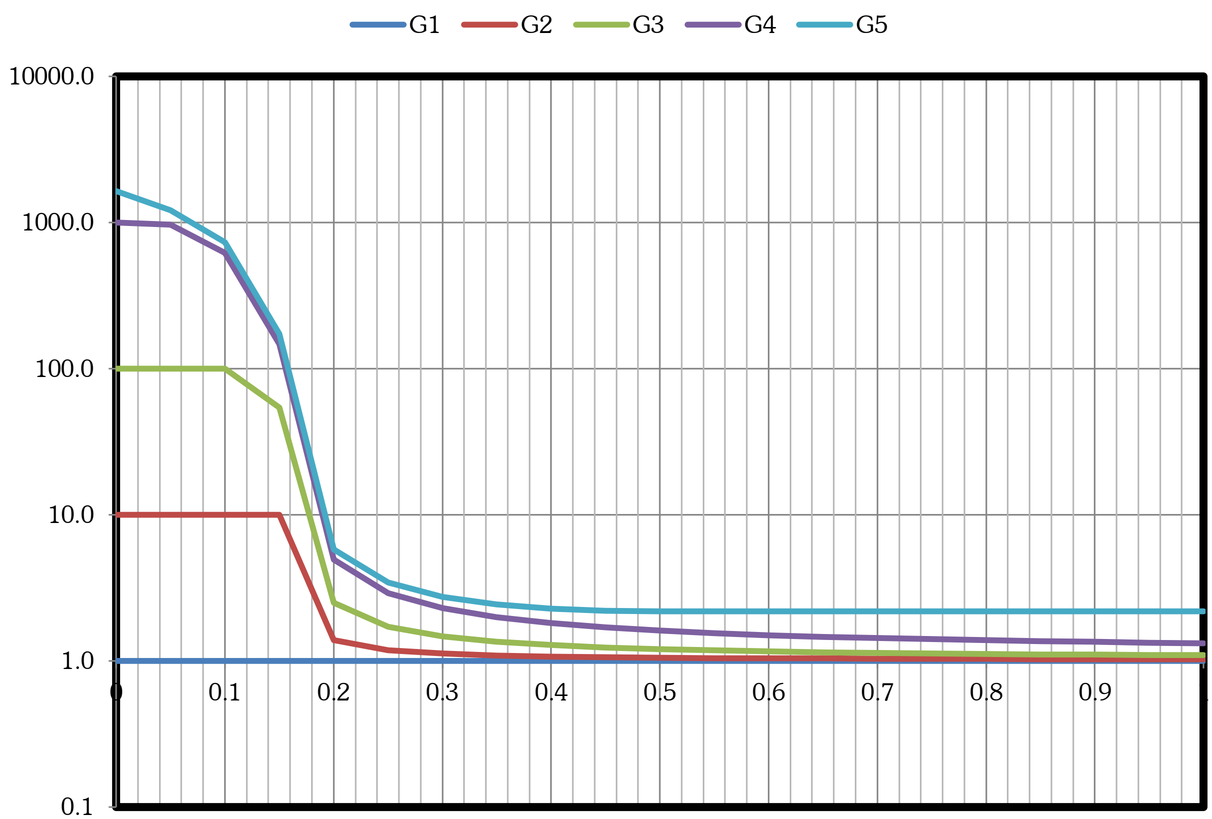
The importance of the transitivity property for the price indexes can be clearly understood when looking at the problem of dealing with N time intervals, t_i, with i = 1, 2, …., N. Without the transitivity property we would have to deal with N(N-1)/2 separate binary comparisons (assuming that at least the reciprocity property holds!). If the transitivity property holds, we can select an arbitrary time interval, say t_r, as the *reference time interval* and then only deal with the N index functions $I(\mathbf{p_i}, \mathbf{p_r}, u \mid U)$. Interestingly enough, we could actually select an arbitrary *reference price vector*, $\mathbf{p_r}$, even although it may not be associated with *any* time interval. Such generalization, however, appears to be more confusing than useful, unless there was some significant justification for a "special" price vector.

In order to understand better the dependence on the choice of the value u, at which to determine the price index, we will return to our deliberately extreme example of section I. If we apply our approach to the situation described by equations (11)-(15) we find that a "good" choice for the parameter vectors **a** and **c** are

(64) $\mathbf{a^o} = (0.2317, 383.0719)$

(65) $\mathbf{c^o} = (3.60504, 0.00043)$

Fig. 1 Index Values as a function of utility value



and that the *optimal* values of the utility function associated with the five time points are:

(66) $$u_1 = 0.4576 \quad u_2 = 0.4466 \quad u_3 = 0.4139 \quad u_4 = 0.3363 \quad u_5 = 0.2980$$

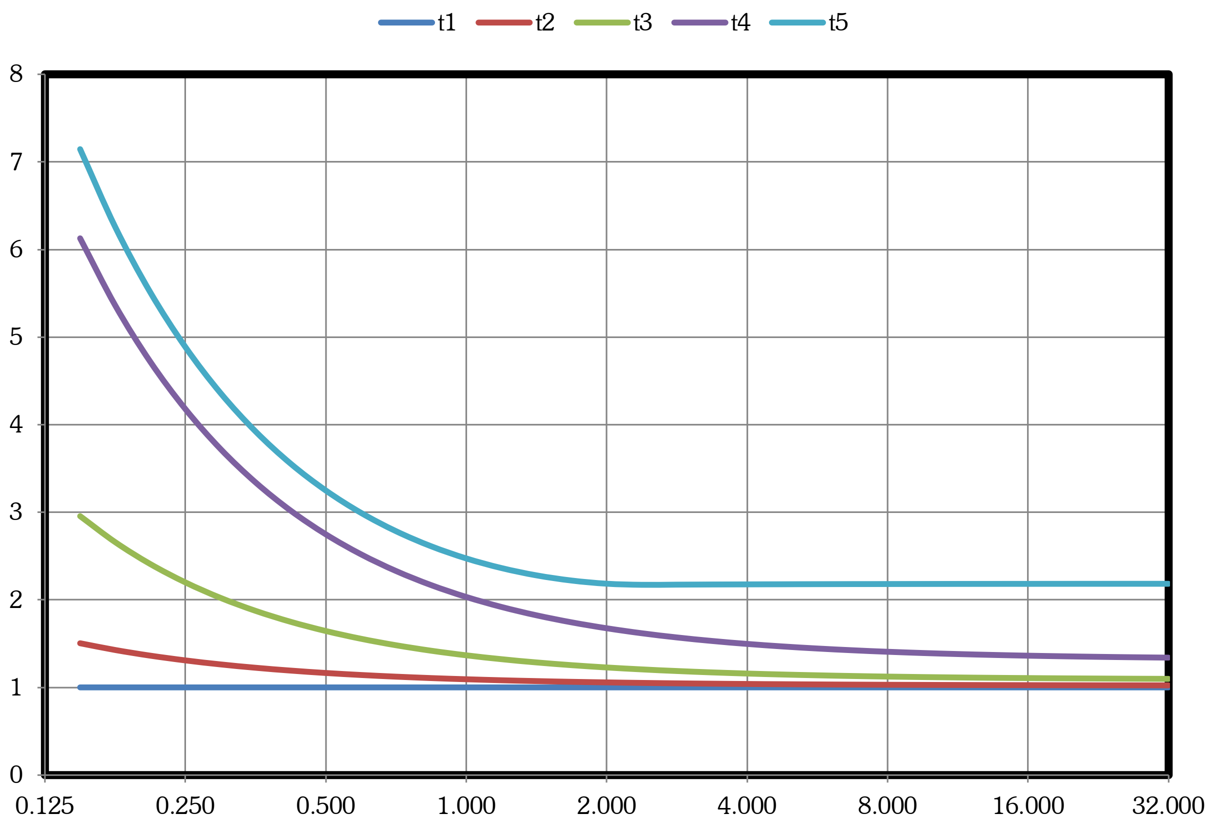
We will choose t_1 as our reference time interval.

In Fig 1 we show the value of the *price index functions* $G_i(u)$ where

(67) $$G_i(u) = I(\mathbf{p_i}, \mathbf{p_1}, u \,|\, \mathbf{a^o}, \mathbf{c^o})$$

Note that index values are shown on a logarithmic scale in order to demonstrate the large differences between the values for small values of u and large values of u.

Fig. 2 Equivalent Budget ratio at different time points



The same basic information is shown in Fig 2 from a different perspective, i.e. as the *equivalent relative budget functions* $B_i(b)$ where

(68) $$B_i(b) = B_{er}(\mathbf{p_i}, \mathbf{p_1}, b \,|\, \mathbf{a^o}, \mathbf{c^o})$$

Both curves show that for low values of u and b (which are related), the price index and the relative equivalent budget are very high, but that they both converge toward much lower values for higher values of u and b. The two sets of curves clearly show that the "value of money" at any given time is not a constant, but a function of the budget of the purchaser.

Let's now go back to the general case. Can we "fairly" summarize the situation with a *single price index value*? Quite obviously we can define an "average value of money' in a variety of ways. For example, let

(69) $$u_i^* = U^*(b_i, \mathbf{p_i} \mid \mathbf{a^o}, \mathbf{c^o})$$

be the optimal value of the utility function that would have been achieved at time t_i according to the assumed choice of parameter vectors $\mathbf{a^o}$ and $\mathbf{c^o}$. Define the *(geometric) average utility* $\bar{u}^*$ as

(70) $$\bar{u}^* = \sqrt[N]{\prod_{i=1}^{N} u_i^*}$$

We can then define the price index $G_1(i, j)$ as

(71) $$G_1(i, j) = I(\mathbf{p_i}, \mathbf{p_j}, \bar{u}^* \mid \mathbf{a^o}, \mathbf{c^o})$$

The price index $G_1(i, j)$ essentially measures the price index at the utility level that corresponds to the geometric average of the utility values that would have been achieved in each time period if the budget available at each time period were optimally utilized, according to the specified choice of parameter vectors for the utility function.

Another approach is to define the index $G_2(i, j)$ as

(72) $$G_2(i, j) = \sqrt[N]{\prod_{k=1}^{N} I(\mathbf{p_i}, \mathbf{p_j}, u_k^* \mid \mathbf{a^o}, \mathbf{c^o})}$$

In the case of the index $G_1(i, j)$ we have taken the geometric average of the utility values and then evaluated the index at that average. In the case of the index $G_2(i, j)$ we have evaluated the indices for all optimal values of the utility function and then we have taken the geometric average of the result. The choice of the *geometric* average for the index $G_2(i, j)$ is suggested by the fact that in this way the *transitivity*, *reciprocity* and *identity* properties will all remain valid. In the case of the index $G_1(i, j)$ this would occur no matter what kind of average we would have chosen. The selection of the geometric average for the $G_1(i, j)$ price index is purely for symmetry with respect to the index $G_2(i, j)$.

In addition to some kind of averaging of the data, there is another approach that also merits attention. This is to select one *fixed* value of u. The selection of a predetermined value for u has an advantage, namely that the price index values do not have to be reevaluated as new data becomes available, for additional time intervals. An interesting choice is to select

(73) $$u = 0.5$$

that corresponds to half of the maximum value of the utility function. Another interesting choice could be to select

(74) $$u = 1$$

i.e. to measure the price index at the "ultimate' value of the utility function. In the latter case it should be noted that the function $B^*(u, \mathbf{p} \mid \mathbf{a}, \mathbf{c})$ goes to infinity as u approaches 1. However, the value of the ratio defined by equation (33) can be evaluated by standard limiting procedure to be

(75) $$\lim_{u \to 1} I(\mathbf{p_x}, \mathbf{p_y}, u \mid \mathbf{a}, \mathbf{c}) = \left[\frac{\sum_m c_m \sqrt{a_m p_{xm}}}{\sum_m c_m \sqrt{a_m p_{ym}}} \right]^2$$

However, the choice of such a limiting case will lead to possibly anomalous results in the presence of very large changes in commodity prices. More generally, we can choose an arbitrary set of values of u and then perform the geometric averaging of the results, as we have done in equation (72). Let

(76) $$\{v_1, v_2, ..., v_n\}$$

be a set of n constants where

(77) $$n \geq 1; \quad 0 < v_k < 1 \quad \text{for all } k$$

We can define the *generalized price index function* $G(i, j \mid v_1, v_2, ..., v_n)$ as

(78) $$G(i, j \mid v_1, ..., v_n) = \sqrt[n]{\prod_{k=1}^{n} I(\mathbf{p_i}, \mathbf{p_j}, v_k \mid \mathbf{a^o}, \mathbf{c^o})}$$

We will define the following particular choices of indexes:

(79) $$G_1(i, j) = G(i, j \mid \overline{u}^*)$$

(80) $$G_2(i, j) = G(i, j \mid u_1^*, u_2^*, ..., u_N^*)$$

(81) $$G_3(i,j) = G(i,j\,|\,0.5)$$

Note that the G_1 and G_2 indexes correspond to the same indexes defined in equations (71) and (72).

If we apply the above choices to our numerical example we will obtain the numbers in Table 1 where we have also added the corresponding values of the Fisher standard index and Fisher chain index. It is clear that the G_1 and G_2 indexes lead to numbers that (at least in this example) are almost identical. Since the value of $\bar{u}^*$ is approximately 0.39, it is not surprising that also the G_3 index (corresponding to $u = 0.5$) is very close. The most interesting fact is however, that all indexes appear to be much more "reasonable" from an intuitive point of view than either the Fisher standard or chain indexes (although the author intuition might be biased).

	i =1, j =1	**i = 2, j = 1**	**i =3, j = 1**	**i = 4, j = 1**	**i = 5, j = 1**
$F(i,j)$	**1.00**	**2.66**	**7.49**	**22.95**	**70.71**
$F_c(i,j)$	**1.00**	**2.66**	**5.03**	**7.10**	**8.55**
$G_1(i,j)$	**1.00**	**1.07**	**1.30**	**1.85**	**2.30**
$G_2(i,j)$	**1.00**	**1.08**	**1.31**	**1.88**	**2.36**
$G_3(i,j)$	**1.00**	**1.05**	**1.20**	**1.61**	**2.17**

Table 1.

V. A Real Example

The trivial example we have been using up to now was meant only to demonstrate a specific difference between our proposed methodology and the classical Fisher approach. We will now look at a real life example in order to have a more valid comparison.

In all of the previous discussion we referred to a single individual purchaser. Unfortunately most of the economic data that we have is *aggregate* data about the overall purchases of all of the US residents. In order to be able to use that data in our analysis we will make the following simplifying assumptions:

- every US purchaser behaves according to the same utility function;
- the overall behavior of the US residents as a whole can be evaluated as if each resident had exactly the same budget, equal to the average budget obtained by dividing the overall budget for the US by the resident population of the US.

This is clearly an oversimplification, but it allows us to analyze some of the available data in a simple way. In order to test the approach in a real life situation we decided to apply it to the US data for the period 1950–2010. The data we used is that released by the Bureau of Economic Analysis (BEA) of the US Department of Commerce, as of June 24, 2011. More precisely, we used the annual BEA Tables labeled "Table 1.5.5 Gross Domestic Product, Expanded Detail" and "Table 1.5.4 Price Indexes for Domestic product, Expanded Detail" in the format made available by the BEA through their Internet website.

Table 2

1	Motor vehicles and parts
2	Furnishings and durable household equipment
3	Recreational goods and vehicles
4	Other durable goods
5	Food and beverages purchased for off-premises consumption
6	Clothing and footwear
7	Gasoline and other energy goods
8	Other nondurable goods
9	Housing and utilities
10	Health care
11	Transportation services
12	Recreation services
13	Food services and accommodations
14	Financial services and insurance
15	Other services
16	Final consumption expenditures of nonprofit institutions serving households
17	Private Investments: Structures
18	Private Investments: Computers and peripheral equipment
19	Private Investments: Software
20	Private Investments: Other office equipment
21	Private Investments: Industrial equipment
22	Private Investments: Transportation equipment
23	Private Investments: Other equipment
24	Private Investments: Residential
25	Federal Defense Consumption expenditures
26	Federal Defense Gross investment
27	Federal Civilian Consumption expenditures
28	Federal Civilian Gross investment
29	State and local Consumption expenditures
30	State and local Gross investment

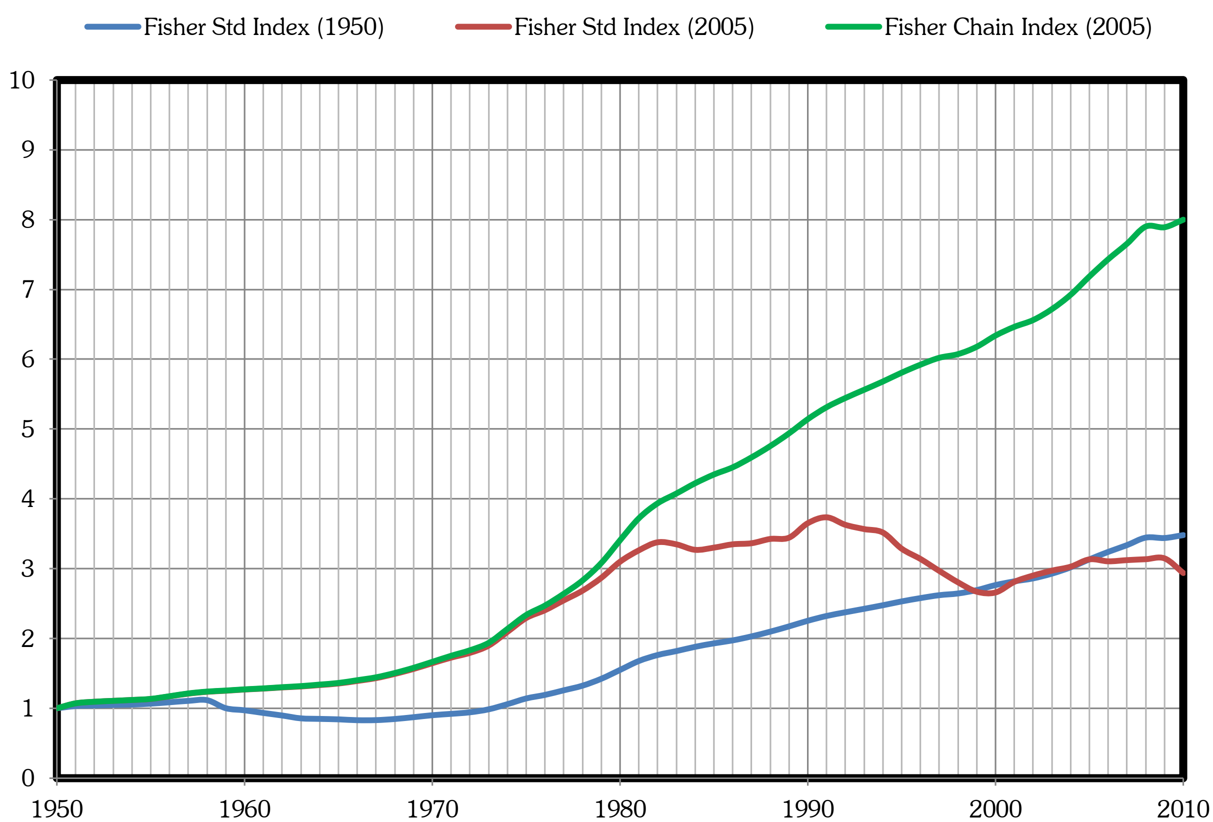
These tables provide the current dollar value of a number of line items and the *relative price* of those items with respect to the *average* of 2005. It is not possible to derive from those tables the *absolute quantities* of the items in question, but only the *relative quantities* with respect to the 2005 average. This is not a major obstacle, since it only changes the units in which all quantity variables are expressed, without affecting the substance of the analysis.

The tables are expressed in units of billion US dollars, with the precision of a tenth of a billion. Because some of the partial aggregate figures were derived from the original (more accurate) data, we restated the total so as to match the available resolution. The data from the tables were modified to be consistent with our model, which is defined as a "purchaser behavior model". We define the **Gross Domestic Activity (GDA)** as the Gross Domestic Product with the following modifications:

- it adds Imports
- it subtracts Exports
- it subtracts any change in inventory

The GDA measures the total amount of money that people have used to purchase goods and services (directly or indirectly) either for immediate consumption or to invest in new assets. With the above modifications, the table identifies the 30 commodities listed in Table 2 which we have used in our analysis. As mentioned above, the original price data is given relative to the year 2005. For our analysis it is preferable to use the year 1950 as the base year. This does not create any problems with either the Fisher chain index or our G indexes. However, as stated earlier, the Fisher standard index does depend on what year is used as the base year.

Fig. 3 Comparison of Fisher Indexes

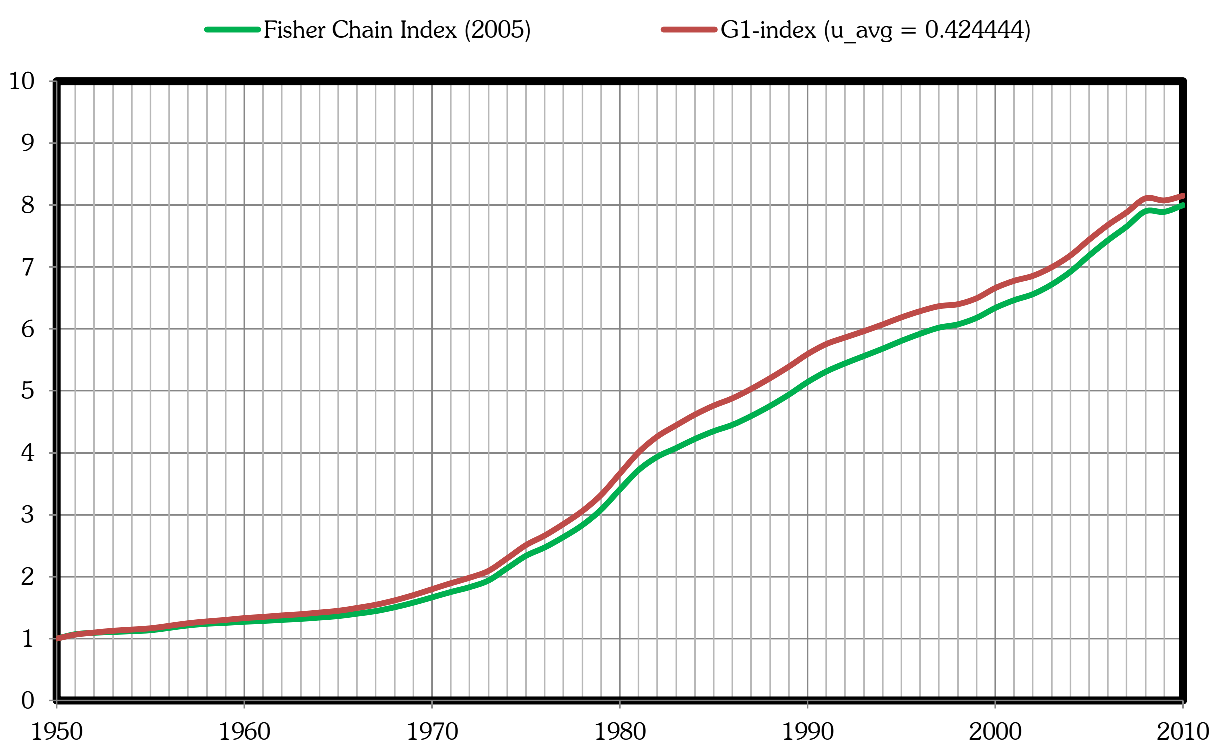


In Fig.3 we show

- the Fisher standard index using the year 1950 as base;
- the Fisher standard index using the year 2005 as base, renormalized so as to the set the value for 1950 to 1;
- the Fisher chain index using the year 1950 as base.
-

The difference between the Fisher standard indexes is quite significant, demonstrating the basic weakness of that index. The apparently "anomalous" behavior of the Fisher standard indexes is due almost exclusively to the presence of the commodity labeled "Private Investment: Computer and Peripheral Equipment". It is a well-known fact that the price of computer units has been going *down* in current $ terms while at the same time their "performance" has been increasing. The BEA has attempted to take this into account, according to the methodology normally referred as "hedonic indexes". Such methodology attempts to restate unit prices so as to take into consideration major changes in functionality. However, in so doing the BEA has gone overboard, overestimating the increased value of such commodity. This author has spent many years in the analysis of computer performance and price/performance, therefore he has some expertise in the subject. We have therefore restated the relative prices of the given commodity to be more in line with reality.

Fig. 4 Comparison between Fisher Chain Index and the G1 Index

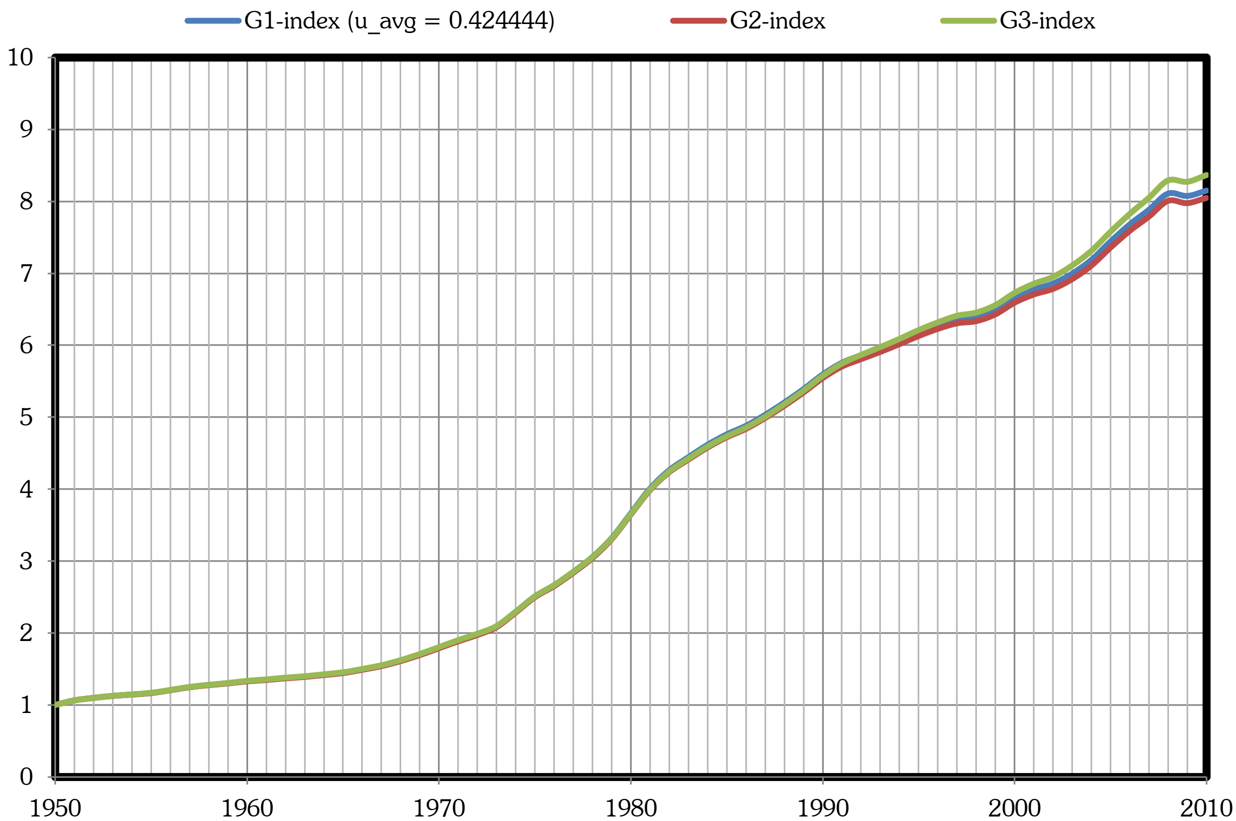


Even so, the still great historical improvement in the price/performance of computers and peripheral equipment distorts completely the Fisher indexes. Since the index uses the prices at one time in combination with the quantities at any other time, it creates an anomalously high evaluation of the index when comparing years at the two extremes of the time period in question.

In Fig.4 we show the Fisher chain index together with the G_1 index of our methodology. They show some significant similarity. This in a sense provides a "degree of confirmation" for the validity of both methodologies, since they achieve a high degree of mutual consistency, while approaching the problem from rather different points of view.

In Fig.5 we show all three of our indexes proposed above. They show remarkable similarity.

Fig.5 Comparison between different G Indexes



Our evaluations have been based on the behavior of an *average* consumer. We know of course that consumers at different income levels will behave differently. However, it is not unreasonable to presume that *all* consumers can be characterized by the same utility function, with the differences in consumption being determined only by the differences in budgets.

In Fig.6 we show the indexes relative to three choices of fixed values of u, namely

(82) $$u = 0.3 \quad u = 0.5 \quad u = 0.7$$

Fig. 6 Index function at different utility values

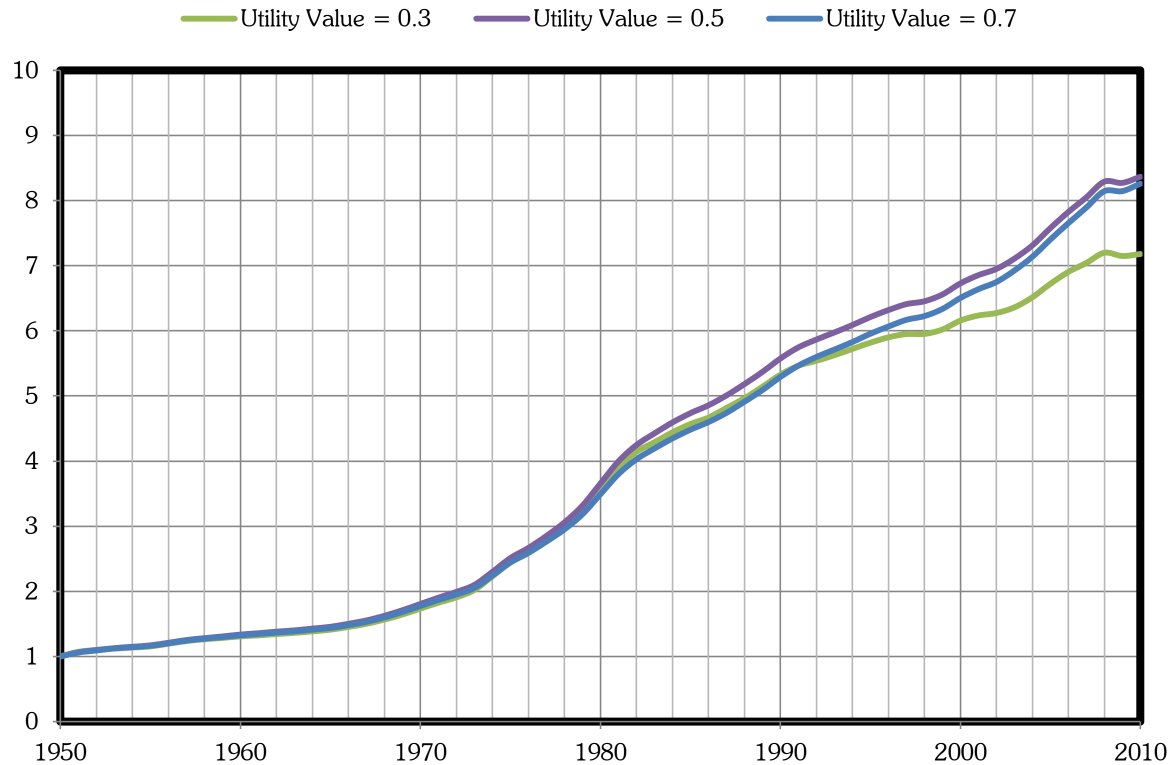
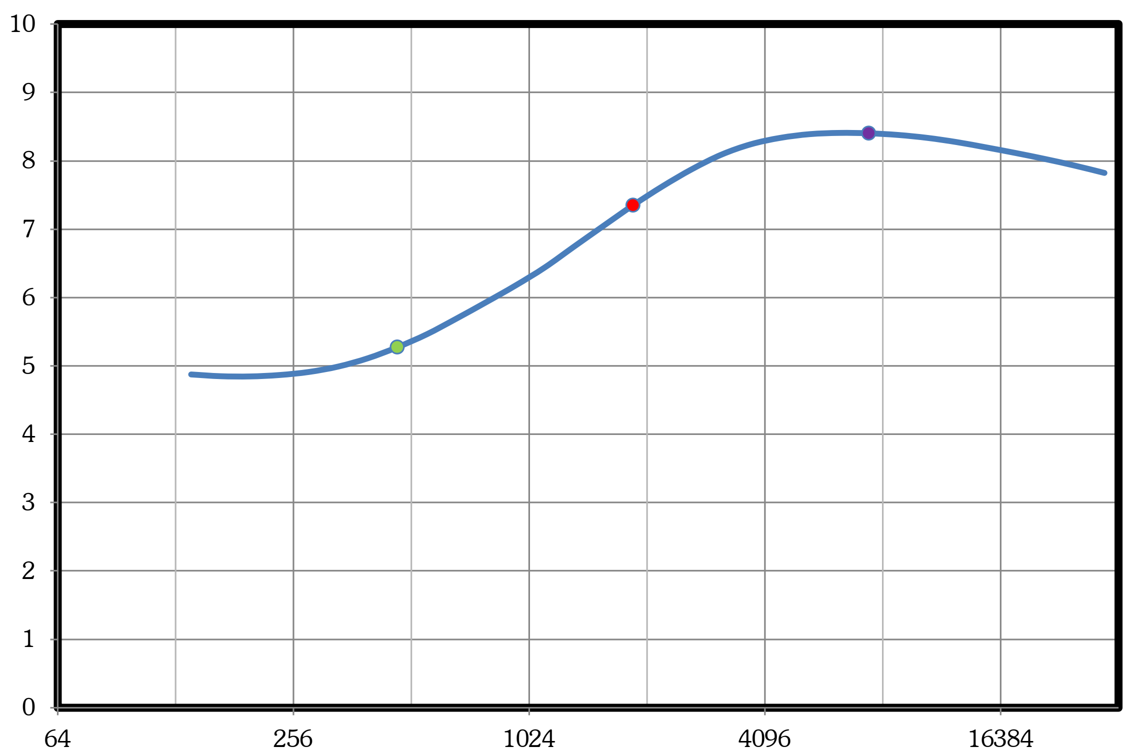


Fig. 7 Equivalent 2010 budget relative to 1950 budget



They may be assumed to approximately correspond to a relatively "low budget", "medium budget" and "high budget" consumer, since budget and utility values would be positively correlated with each other. If we limit ourselves to only look at the two extremes of the time period, i.e. the years 1950 and 2010, Fig.7 shows the equivalent budget ratio that would have been necessary to obtain in the year 2010 in order to achieve the same level of utility (in 2010) achieved by any given budget level in 1950. For the average budget (in 1950) the ratio is about 7.3, for a budget equal of one fourth of the average (in 1950), the ratio is about 5.2. For a budget equal to 4 times the average (in 1950) the ratio is about 8.4.

VI. Time Varying Utility Function

In the previous analysis we have assumed that the utility function is constant over time. When the overall time span is significant, this assumption may be questioned. We could approach the issue by hypothesizing a specific form of time dependence for the **a** and **c** vectors. Unfortunately the variety of reasonable options would lead us into a considerable level of complexity that may be inappropriate in the present context.

There is however a simple approach that can give us some insight into the issue. Assume we have N time intervals, t_i, with i = 1, 2, …., N. Let r be an integer, with $1 \le r \le N$. For each value of the index j, $1 \le j \le N$, define

(83) $$j_L = \max(0, j - r)$$

(84) $$j_H = \min(j + r, N)$$

In other words the interval (j_L, j_H) consists normally of $2r+1$ time points, centered around the time interval t_j, as long as such time points are available. Define the *local error function*

(85) $$S(j,\mathbf{a},\mathbf{c}) = \sum_{i=j_L}^{j_H} \frac{1}{b_i^2} \left[\sum_{m=1}^{M} \left[p_{im} \left[q_{im} - q_m^*(b_i, \mathbf{p_i} \mid \mathbf{a}, \mathbf{c}) \right] \right]^2 \right]$$

which only takes into consideration the differences between the actual and optimal values of the quantity vectors for a limited number of time intervals, centered on the selected time interval. Define the *locally optimal vectors* $\mathbf{a^o}(j)$ and $\mathbf{b^o}(j)$ as those parameter vectors that minimize the function $S(j,\mathbf{a},\mathbf{c})$, *subject to the constraint*

(86) $$\mathbf{q_j} = \mathbf{q_j^*}\left(b_j, \mathbf{p_j} \mid \mathbf{a^o}(j), \mathbf{c^o}(j)\right)$$

i.e. subject to the constraint that for the specified index *j* the values of the *actual* quantity vector and of the *optimal* quantity vector are exactly the same. In other words, the utility function $U\left(\mathbf{q}\,|\,\mathbf{a^o}(j),\mathbf{c^o}(j)\right)$ is biased to match the actual purchasing behavior at time t_j, while minimizing the differences for the errors in neighboring time intervals. Let

(87) $$u_i^*(j) = U^*(b_i,\mathbf{p_i}\,|\,\mathbf{a^o}(j),\mathbf{c^o}(j))$$

and

(88) $$\bar{u}(j) = \sqrt[N]{\prod_{i=1}^{N} u_i^*(j)}$$

We can then define the analog indexes to the 3 introduced in equations (79) - (81), i.e.

(89) $$G_1(i,j\,|\,k,r) = I(\mathbf{p_i},\mathbf{p_j},\bar{u}^*(k)\,|\,\mathbf{a^o}(k),\mathbf{c^o}(k))$$

(90) $$G_2(i,j\,|\,k,r) = \sqrt[N]{\prod_{s=1}^{N} I(\mathbf{p_i},\mathbf{p_j},u_s^*(k)\,|\,\mathbf{a^o}(k),\mathbf{c^o}(k))}$$

(91) $$G_3(i,j\,|\,k,r) = I(\mathbf{p_i},\mathbf{p_j},0.5\,|\,\mathbf{a^o}(k),\mathbf{c^o}(k))$$

as the *locally optimal price indexes relative to the time interval k*.

We can apply the proposed approach to the example discussed in the previous section. We selected

(92) $$r = 10$$

for our analysis.

In order to make the figures somewhat more readable, we only show in Fig.8 the G_3 indexes for the years 1950, 1960, 1970,….. 2000, 2010. There is a noticeable systematic difference, with the utility functions optimized for later years indicating a higher rate of inflation. In this study we are concentrating on the methodology, so we will not attempt to analyze the reasons for the trend.

We can combine the locally optimal indexes into overall indexes

(93) $$\bar{G}_m(i,j\,|\,r) = \sqrt[n]{\prod_{k=1}^{n} G_m(i,j\,|\,k,r)} \qquad \text{for all } m$$

where n is the number of time intervals for which the local optimization has been performed.

Fig. 8 Yearly optimized indexes

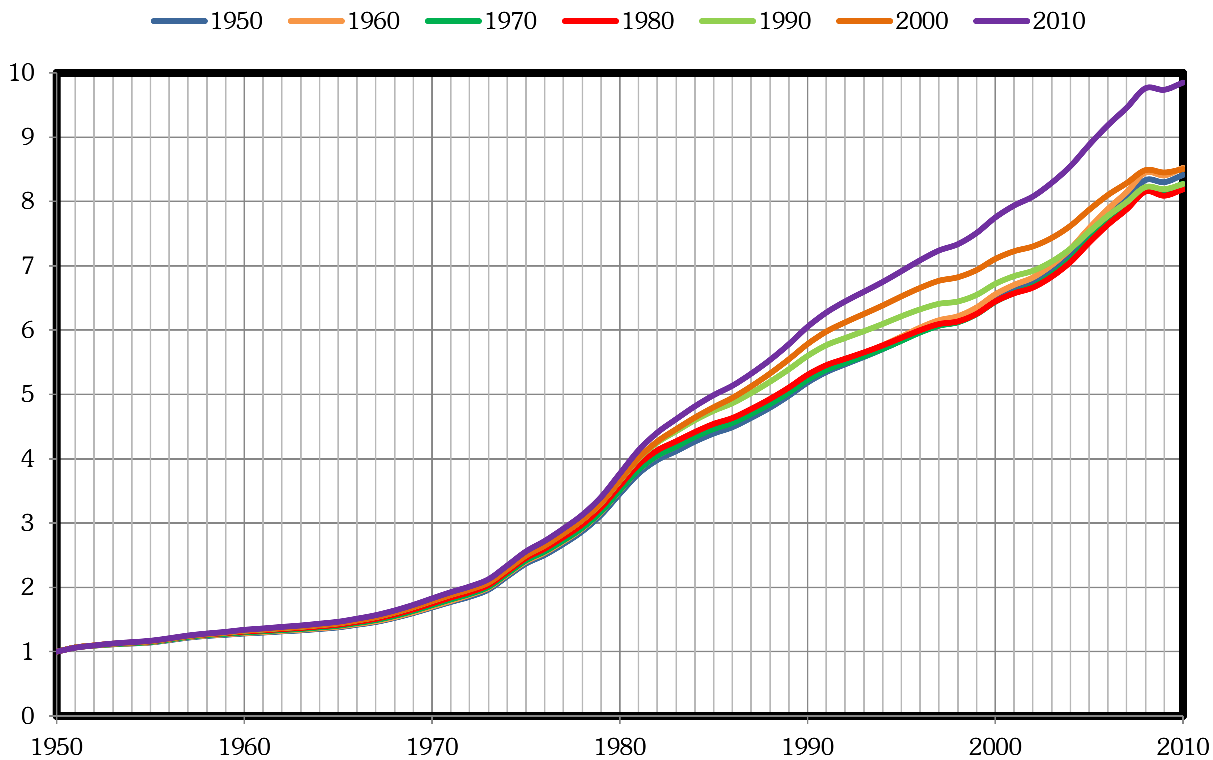
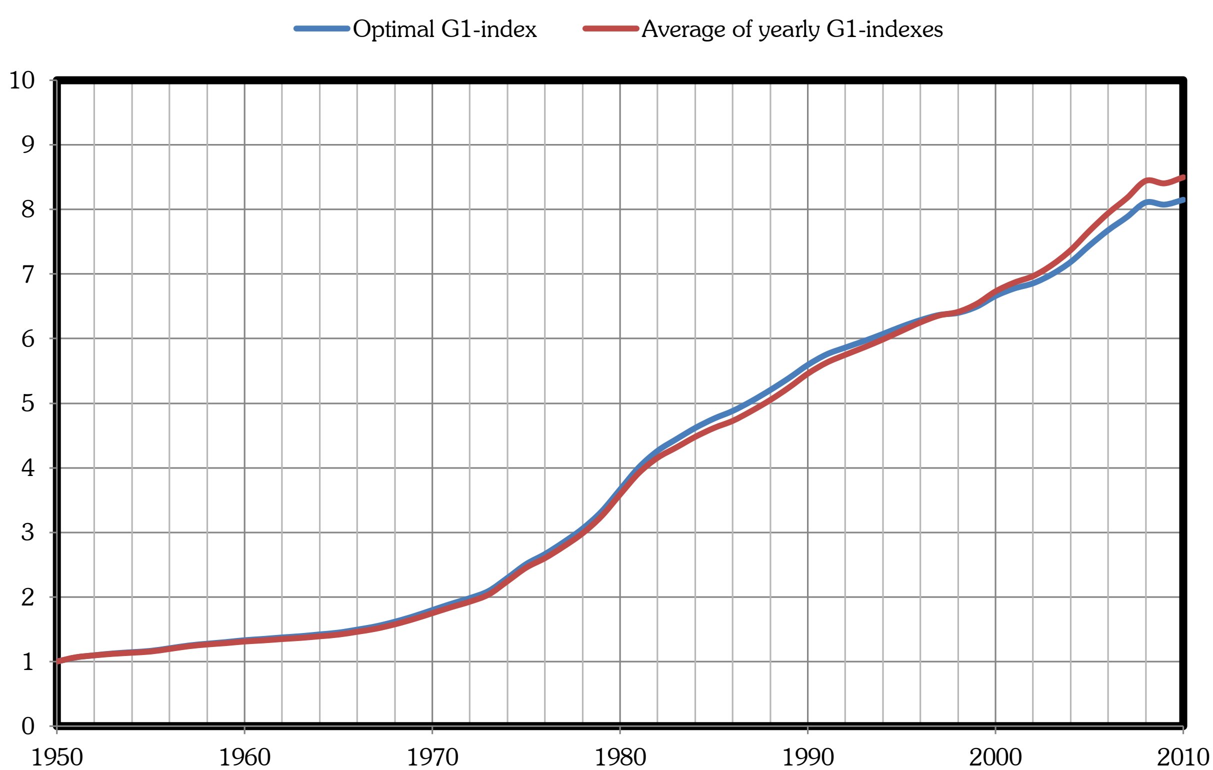


Fig. 9 Comparison of G1 indexes



In fig.9 we show the value of the $\bar{G}_3$ index together with the G_3 index associated with the constant U, showing considerable agreement between the two.

VII. Year Pairs Analysis

In our original analysis discussed in section V we have taken simultaneously into account all of the available data for the period 1950-2010. In section VI we have used part of the data in order to determine each of the locally optimal utility functions, but then we have applied each of those utility functions to the whole time interval. We will now look at the data in a different way. We select one time interval, say t_r as the *reference time interval*. For each time interval t_i we will determine two sets of function parameter vectors $(\mathbf{a}^{\mathbf{o}}_{\mathbf{ir}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ir}})$ and $(\mathbf{a}^{\mathbf{o}}_{\mathbf{ri}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ri}})$ each of which minimizes the error function

$$(94)\; S(i,r,\mathbf{a},\mathbf{c}) = \frac{1}{b_i^2}\left[\sum_{m=1}^{M}\left[p_{im}\left[q_{im}-q_m^*\left(b_i,\mathbf{p_i}\,|\,\mathbf{a},\mathbf{c}\right)\right]\right]^2\right] + \frac{1}{b_r^2}\left[\sum_{m=1}^{M}\left[p_{rm}\left[q_{rm}-q_m^*\left(b_r,\mathbf{p_r}\,|\,\mathbf{a},\mathbf{c}\right)\right]\right]^2\right]$$

subject to the constraint, in the case of the first set, that the actual and optimal quantity vectors are exactly equal for time interval t_r, and for the second set that the corresponding vectors are exactly equal for time interval t_i. It should be noted that in the case of only a pair of time intervals it may be possible to find algebraically a pair of parameter vectors such that

$$(95) \qquad S(i,r,\mathbf{a},\mathbf{c}) = 0$$

However, we will use the same gradient technique used in the general case, so as to maintain the general consistency of methodology.

We can then define the *optimal time interval pair G3 Index* as

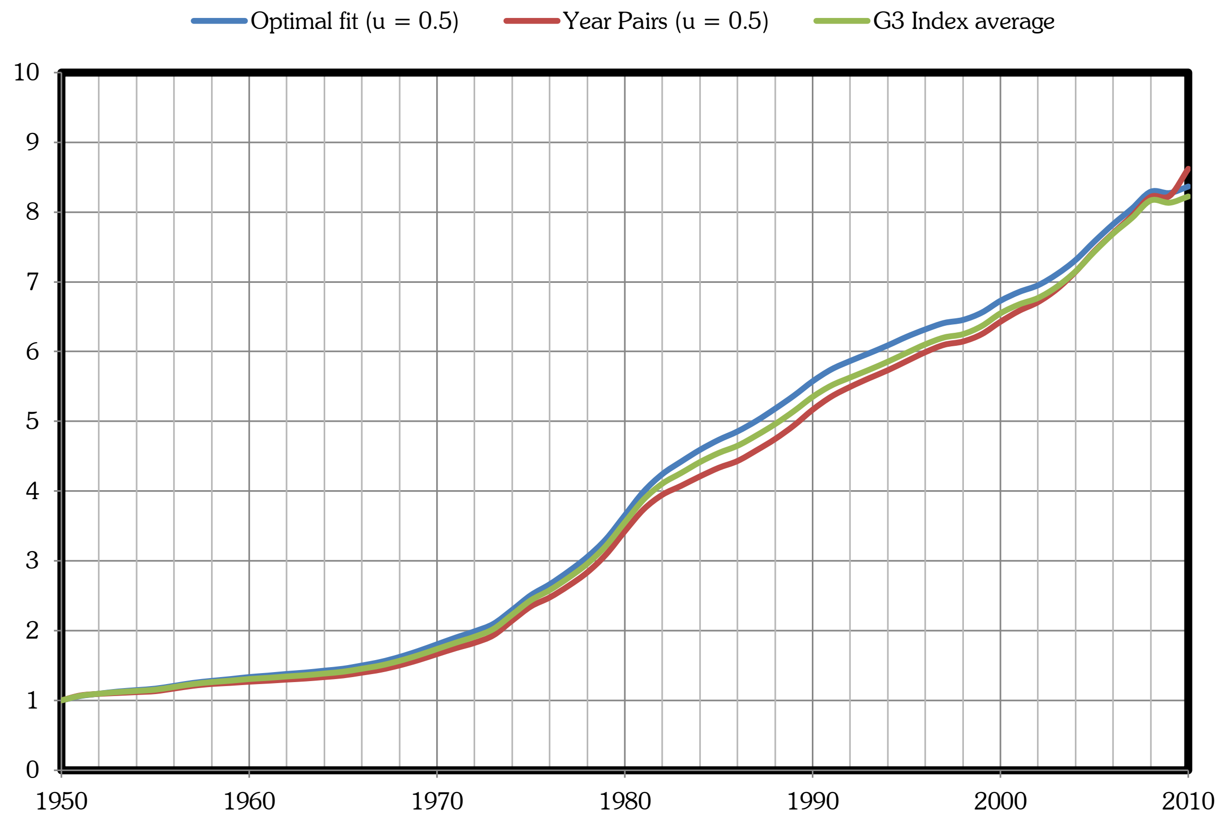
$$(96) \qquad \bar{G}_{3p}\left(i\,|\,r\right) = \sqrt{I(\mathbf{p_i},\mathbf{p_r},0.5\,|\,\mathbf{a}^{\mathbf{o}}_{\mathbf{ir}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ir}}) \times I(\mathbf{p_i},\mathbf{p_r},0.5\,|\,\mathbf{a}^{\mathbf{o}}_{\mathbf{ri}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ri}})}$$

Note that the above can also be written as

$$(97) \qquad \bar{G}_{3p}(i\,|\,r) = \sqrt{\frac{B^*(0.5,\mathbf{p_i}\,|\,\mathbf{a}^{\mathbf{o}}_{\mathbf{ir}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ir}})}{B^*(0.5,\mathbf{p_r}\,|\,\mathbf{a}^{\mathbf{o}}_{\mathbf{ir}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ir}})} \times \frac{B^*(0.5,\mathbf{p_i}\,|\,\mathbf{a}^{\mathbf{o}}_{\mathbf{ri}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ri}})}{B^*(0.5,\mathbf{p_r}\,|\,\mathbf{a}^{\mathbf{o}}_{\mathbf{ri}},\mathbf{c}^{\mathbf{o}}_{\mathbf{ri}})}}$$

It should be noted a formal analogy with the Fisher standard index. The key difference is that instead of using the "theoretical budget" associated with the quantities *actually purchased* at the appropriate time intervals, the index uses the *optimal* budgets associated with the predetermined value of the utility function, evaluated at the appropriate price vector. Both the Fisher standard index and the $\bar{G}_{3p}(i\,|\,r)$ defined above have the characteristic that only the data for the two specified time intervals are used, *without any reference to any intervening time intervals*.

Fig. 10



In fig.10 we show three indexes, all evaluated at the utility value of 0.5: the G_3 index defined in section IV, the $\bar{G}_3(i,1950\,|\,10)$, already shown in Fig.9 and the above defined $\bar{G}_{3p}(i\,|\,r)$. All three indexes are in very good agreement. The important fact to note here is that while the first two indexes were evaluated on the basis of the full knowledge of the whole information in the 1950-2010 period, each point of the last index was evaluated using only the information about the two time period in question. In other words, in our methodology, by using the $\bar{G}_{3p}(i\,|\,r)$ index defined above it is possible to evaluate a realistic price index for any two time intervals, without any knowledge of any intervening data, *no matter how distant the two time intervals are*. We believe that this demonstrates quite clearly the "robustness" of the approach that we have been discussing.

VIII. Conclusions

We believe that both the Fisher standard index and the Fisher chain index methodologies suffer from very disturbing anomalies and lack any theoretical underpinning. Approaches based on underlying utility functions have been discussed extensively in the theoretical literature, but appear to have had limited applications in actual practice. There are probably two basic reasons for the situation, namely:

- there is often a reluctance to make apparently “arbitrary” assumptions about the general form of the utility function;
- the computational complexity of “fitting” a large set of parameters may have been considered ”impractical”.

We believe that the first objection is not a valid one. The purpose of “models” is to allow us to derive useful criteria for determining a course of action. The “validity” of such models needs to be judged only by their usefulness (or lack thereof). There may have been some validity to the second objection before digital computers, but certainly not now.

The approach that we have presented is characterized by an underlying theoretical foundation that, once accepted, leads in a congruent way to the determination of the value of the indexes. Such indexes meet certain important requirements, namely they satisfy the transitivity, reciprocity and identity properties. Furthermore, the approach shows considerable robustness in the presence of drastic price changes. The approach allows for a variety of choices in the selection of specific indexes. This might be viewed as a drawback. We believe instead that this “embarrassment of riches” points to the fact that it is wishful thinking to believe that a “unique” characterization of the value of money may be found that is valid for any study of the issue. Furthermore, it suggests that attempts to characterize the value of money with high precision are probably misguided.

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